

# JACOB'S LADDERS AND INVARIANT SET OF CONSTRAINTS FOR THE REVERSELY ITERATED INTEGRALS (ENERGIES) IN THE THEORY OF THE RIEMANN ZETA-FUNCTION

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ABSTRACT. In this paper we obtain an extension of the set of non-local equalities by adding to it new set of local equalities. Namely, we obtain an invariant set of equalities on the set of reversely iterated integrals (energies). In other words, we obtain a new continuum set of constraints on behaviour of the function  $\zeta\left(\frac{1}{2} + it\right)$ ,  $t \rightarrow \infty$ .

## 1. INTRODUCTION

1.1. Let us remind we have proved the following theorem (see [4]): for the class  $G(S)$  of functions

$$g = g(u_1, \dots, u_n), \quad (u_1, \dots, u_n) \in S \subset R^n$$

such that

$$(1.1) \quad g \geq 0, \quad g = o\left(\frac{T}{\ln T}\right), \quad T \rightarrow \infty$$

we have the following formula

$$(1.2) \quad \forall g \in G(S) : \int_T^{\widehat{T+g}} \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt = g, \quad k = 1, \dots, k_0$$

for every fixed  $k_0 \in \mathbb{N}$  and for every sufficiently big  $T > 0$ .

*Remark 1.* Let us notice that the level of precision of the formula (1.2) is characterized sufficiently by the following example

$$\int_T^{\widehat{T+10^{-60}}} \prod_{r=0}^{256} \left| \zeta\left(\frac{1}{2} + i\varphi_1^r(t)\right) \right|^2 dt = 10^{-60} \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln T}{\ln T}\right) \right\} \ln^{257} T.$$

This formula is not accessible neither by the Riemann-Siegel formula

$$Z(t) = 2 \sum_{n \leq \bar{t}} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + \mathcal{O}(t^{-1/4}), \quad \bar{t} = \sqrt{\frac{t}{2\pi}},$$

nor by the current methods in the theory of the Riemann zeta-function.

The formula (1.2) appears within the following context.

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*Key words and phrases.* Riemann zeta-function.

## 1.2. The complicated signal

$$\prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)]; \quad \tilde{Z}^2[\varphi_1^0(t)] = \tilde{Z}^2(t)$$

is generated by the primary signal

$$(1.3) \quad \begin{aligned} Z(t) &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \\ \vartheta(t) &= -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right) \end{aligned}$$

which itself is generated by the Riemann zeta-function on the critical line. Namely, in connection with (1.3), we have introduced (see [2], (9.1), (9.2)) the formula

$$(1.4) \quad \tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt},$$

where

$$(1.5) \quad \begin{aligned} \tilde{Z}^2(t) &= \frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]} = \frac{|\zeta(\frac{1}{2} + it)|^2}{\omega(t)}, \\ \omega(t) &= \left\{1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\right\} \ln t. \end{aligned}$$

The function

$$\varphi_1(t)$$

which is called the Jacob's ladder (see our paper [1]) according to the Jacob's dream in Chumash, Bereishis, 28:12, has the following properties:

(a)

$$\varphi_1(t) = \frac{1}{2}\varphi(t),$$

(b) the function  $\varphi(t)$  is solution of the non-linear integral equation (see [1], [2])

$$\int_0^{\mu[x(T)]} Z^2(t) e^{-\frac{2}{x(T)}t} dt = \int_0^T Z^2(t) dt,$$

where each admissible function  $\mu(y)$  generates the solution

$$y = \varphi_\mu(T) = \varphi(T); \quad \mu(y) \geq 7y \ln y.$$

*Remark 2.* The main goal of introducing Jacob's ladders is described in [1], where we have proved by making use of these Jacob's ladders, that the Hardy-Littlewood integral (1918)

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt$$

has – in addition to the Hardy-Littlewood expression (and also other similar to this one) possessing an unbounded error at  $T \rightarrow \infty$  – the following set of almost exact expressions

$$\begin{aligned} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt &= \varphi_1(T) \ln \varphi_1(T) + (c - \ln 2\pi) \varphi_1(T) + \\ &+ c_0 + \mathcal{O}\left(\frac{\ln T}{T}\right), \quad T \rightarrow \infty, \end{aligned}$$

where  $c$  is the Euler's constant, and  $c_0$  is the constant from the Titchmarsh-Kober-Atkinson formula (see [5], p. 141).

*Remark 3.* The Jacob's ladder  $\varphi_1(T)$  can be interpreted by our formula (see [1], (6.2))

$$(1.6) \quad T - \varphi_1(T) \sim (1 - c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T}, \quad T \rightarrow \infty,$$

where  $\pi(T)$  is the prime-counting function, as an asymptotically complementary function to the function

$$(1 - c)\pi(T)$$

in the following sense

$$(1.7) \quad \varphi_1(T) + (1 - c)\pi(T) \sim T, \quad T \rightarrow \infty.$$

*Remark 4.* Let us notice explicitly that the main reason for the substitution

$$\frac{T}{\ln T} \longrightarrow \pi(T), \quad T \rightarrow \infty$$

by the Hadamard-de la Vallé Poussin formula

$$\pi(T) \sim \frac{T}{\ln T}, \quad T \rightarrow \infty$$

lies in availability of the asymptotic law (1.7) of complementarity of the functions

$$\varphi_1(T), \quad (1 - c)\pi(T), \quad T \rightarrow \infty,$$

(see also [1], (6.1) – (6.6)).

## 2. ON STRUCTURE OF NON-LOCAL EQUALITIES

2.1. Now we turn back to the formula (1.2). It is clear that the following set of equalities

$$(2.1) \quad \begin{aligned} \int_{\widehat{T}}^{\widehat{T+g}} \tilde{Z}^2(t) dt &= \int_{\widehat{T}}^{\widehat{T+g}} \prod_{r=0}^1 \tilde{Z}^2[\varphi_1^r(t)] dt = \dots = \\ &= \int_{\widehat{T}}^{\widehat{T+g}} \prod_{r=0}^k \tilde{Z}^2[\varphi_1^r(t)] dt, \quad k = 1, 2, \dots, k_0 \end{aligned}$$

is contained in (1.2) for every sufficiently big  $T$ . The product

$$(2.2) \quad \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)]$$

in (2.1) has the following properties. First of all we have that (see [3], (2.10))

$$(2.3) \quad \begin{aligned} \varphi_1^0(t) = t \in [\widehat{T}, \widehat{T+g}] &\Rightarrow \varphi_1^r(t) \in [\widehat{T}^{k-r}, \widehat{T+g}^{k-r}], \\ r = 1, \dots, k, \quad k &\leq k_0, \end{aligned}$$

i.e.

$$\begin{aligned}
 \varphi_1^1(t) &\in [\overset{k-1}{T}, \overset{k-1}{T+g}], \\
 \varphi_1^2(t) &\in [\overset{k-2}{T}, \overset{k-2}{T+g}], \\
 (2.4) \quad &\vdots \\
 \varphi_1^{k-1}(t) &\in [\overset{1}{T}, \overset{1}{T+g}], \\
 \varphi_1^k(t) &\in [\overset{0}{T}, \overset{0}{T+g}] = [T, T+g].
 \end{aligned}$$

Next, we have the following properties for the segments in (2.3), (see [3], (2.5) – (2.7))

$$\begin{aligned}
 g &= o\left(\frac{T}{\ln T}\right) \Rightarrow \\
 (2.5) \quad |[T, \overset{k}{T+g}]| &= \overset{k}{T+g} - T = o\left(\frac{T}{\ln T}\right),
 \end{aligned}$$

$$(2.6) \quad |[T+g, \overset{k}{T}]| = \sim (1-c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T},$$

$$(2.7) \quad [T, T+g] \prec [\overset{1}{T}, \overset{1}{T+g}] \prec \dots \prec [\overset{k}{T}, \overset{k}{T+g}],$$

where, of course, (see (2.6))

$$(2.8) \quad \rho\{\overset{r-1}{T}, \overset{r-1}{T+g}, [\overset{r}{T}, \overset{r}{T+g}]\} \sim (1-c)\pi(T)$$

and  $\rho$  is the distance of corresponding segments.

*Remark 5.* Asymptotic behaviour of the following disconnected set

$$(2.9) \quad \Delta(T, k, g) = \bigcup_{r=0}^k [\overset{r}{T}, \overset{r}{T+g}]$$

(see (2.5)–(2.7), comp. [3], (2.9)) is as follows: if  $T \rightarrow \infty$  then the components of the set (2.9) recede unboundedly each from other and all together are receding to infinity. Hence, if  $T \rightarrow \infty$  then the set (2.9) looks like one dimensional Friedmann-Hubble expanding universe.

*Remark 6.* Consequently, we notice the following:

- (a) the equalities in (2.1) are non-local ones (see (2.7), (2.8)) – this is the property of external non-localization,
- (b) simultaneously, the integrals in (1.2) with  $k = 2, \dots, k_0$  are non-local (see (1.2), (2.2)–(2.4)) – this is the property of internal non-localization.

In this paper we obtain an extension of the set of non-local equalities (2.1) by adding to it a new set of local equalities – i.e. equalities without the property (a) – for the reversely iterated integrals (energies). In other words, we obtain an invariant set (at  $T \rightarrow \infty$ ) of equalities-constraints on the set of reversely iterated integrals (energies).

## 3. THEOREM ON A SET OF LOCAL EQUALITIES

3.1. Let us remind (see [3], (5.2), (5.7)) that

$$(3.1) \quad T < \overset{1}{T} < \overset{2}{T} < \dots < \overset{k-1}{T}, \quad k = 2, \dots, k_0,$$

where

$$(3.2) \quad \varphi_1(\overset{k}{T}) = \overset{k-1}{T}.$$

Next, putting

$$T \longrightarrow \overset{1}{T}$$

in (3.1) we obtain that

$$(3.3) \quad \overset{1}{T} < \overset{2}{T} < \dots < \overset{k}{T}.$$

Now, let us consider the segment (see (1.2))

$$[\overset{1}{T}, \overset{1}{T+g}]$$

and assign to it (for every fixed  $g \in (0, \infty)$ ) the following sequence of segments

$$(3.4) \quad [\overset{2}{T}, \overset{1}{T+g}], [\overset{3}{T}, \overset{2}{T+g}], \dots, [\overset{k}{T}, \overset{k-1}{T+g}],$$

and, consequently, to each of these segments we assign the integral (energy)

$$[\overset{p}{T}, \overset{p-1}{T+g}] \longrightarrow \int_{\overset{p}{T}}^{\overset{p-1}{T+g}} \tilde{Z}^2(t) dt, \quad p = 2, \dots, k.$$

Since

$$\int_{\overset{1}{T}}^{\overset{1}{T+g}} \tilde{Z}^2(t) dt = g$$

for every sufficiently big  $T$ , then (see (3.2), (3.3))

$$(3.5) \quad \begin{aligned} \int_{\overset{1}{T}}^{\overset{1}{T+g}} \tilde{Z}^2(t) dt &= \int_{\overset{1}{T}}^{\overset{p-1}{T+g}} \tilde{Z}^2(t) dt = \\ &= \int_{\overset{p}{T}}^{\overset{p-1}{T+g}} \tilde{Z}^2(t) dt = g, \quad p = 2, \dots, k. \end{aligned}$$

Hence, we have the following formula (see (1.2), (3.5))

$$(3.6) \quad \int_{\overset{p}{T}}^{\overset{p-1}{T+g}} \tilde{Z}^2(t) dt = \int_{\overset{p}{T}}^{\overset{p}{T+g}} \prod_{r=0}^{p-1} \tilde{Z}^2[\varphi_1^r(t)] dt = g, \quad p = 2, \dots, k.$$

3.2. By the similar way we obtain

$$\begin{aligned}
 [T, \overbrace{T}^{p-2} + g] &\longrightarrow \int_T^{\overbrace{T}^{p-2} + g} \prod_{r=0}^1 = \int_T^{\overbrace{T+g}^p} \prod_{r=0}^{p-1} = g, \quad p = 3, \dots, k, \\
 [T, \overbrace{T}^{p-3} + g] &\longrightarrow \int_T^{\overbrace{T}^{p-3} + g} \prod_{r=0}^2 = \int_T^{\overbrace{T+g}^p} \prod_{r=0}^{p-1} = g, \quad p = 4, \dots, k, \\
 (3.7) \quad &\vdots \\
 [T, \overbrace{T}^{p-k+2} + g] &\longrightarrow \int_T^{\overbrace{T}^{p-k+2} + g} \prod_{r=0}^{k-3} = \int_T^{\overbrace{T+g}^p} \prod_{r=0}^{p-1} = g, \quad p = k-1, k, \\
 [T, \overbrace{T}^{p-k+1} + g] &\longrightarrow \int_T^{\overbrace{T}^{p-k+1} + g} \prod_{r=0}^{k-2} = \int_T^{\overbrace{T+g}^p} \prod_{r=0}^{p-1} = g, \quad p = k.
 \end{aligned}$$

**Theorem.** For every fixed (comp. (1.1))

$$g \in (0, +\infty), \quad k_0 \in \mathbb{N}, \quad k_0 \geq 2$$

there is such

$$T_0[\varphi_1; g] > 0$$

that for every

$$(3.8) \quad T \in (T_0[\varphi_1; g], +\infty)$$

the matrix

$$\begin{aligned}
 (3.9) \quad &\left\| \int_T^{\overbrace{T}^s + g} \prod_{r=0}^{s-1} \tilde{Z}^2[\varphi_1^r(t)] dt \right\|_{p,s}, \\
 &p = s+1, \dots, k, \quad s = 1, \dots, k-1; \quad 2 \leq k \leq k_0
 \end{aligned}$$

of the reversely iterated integrals (energies) fulfills the following equalities

$$\begin{aligned}
 (3.10) \quad & g = \int_T^{\overbrace{T+g}^1} \prod_0^0 \\
 & \parallel \\
 & \int_T^{\overbrace{T+g}^2} \prod_0^0 = \int_T^{\overbrace{T+g}^2} \prod_0^1 \\
 & \parallel \\
 & \int_T^{\overbrace{T+g}^3} \prod_0^0 = \int_T^{\overbrace{T+g}^2} \prod_0^1 = \int_T^{\overbrace{T+g}^3} \prod_0^2 \\
 & \parallel \\
 & \vdots \\
 & \parallel \\
 & \int_T^{\overbrace{T+g}^{k-2}} \prod_0^0 = \int_T^{\overbrace{T+g}^{k-3}} \prod_0^1 = \int_T^{\overbrace{T+g}^{k-4}} \prod_0^2 = \cdots = \int_T^{\overbrace{T+g}^{k-2}} \prod_0^{k-3} \\
 & \parallel \\
 & \int_T^{\overbrace{T+g}^{k-1}} \prod_0^0 = \int_T^{\overbrace{T+g}^{k-2}} \prod_0^1 = \int_T^{\overbrace{T+g}^{k-3}} \prod_0^2 = \cdots = \int_T^{\overbrace{T+g}^{k-2}} \prod_0^{k-3} = \int_T^{\overbrace{T+g}^{k-1}} \prod_0^{k-2}.
 \end{aligned}$$

*Remark 7.* Let us notice that the name *local equalities* we use for every fixed row in the matrix (3.10) only. Namely, the corresponding segments

$$\begin{aligned}
 & \overbrace{[T, T+g]}^1, \quad \overbrace{[T, T+g]}^2, \quad \overbrace{[T, T+g]}^3, \\
 & \overbrace{[T, T+g]}^4, \quad \overbrace{[T, T+g]}^3, \quad \overbrace{[T, T+g]}^2, \quad \overbrace{[T, T+g]}^1,
 \end{aligned}$$

for example, get on from the same point. Simultaneously, the distance of every two consecutive rows is (comp. (2.8))

$$\sim (1-c)\pi(T) \sim (1-c)\frac{T}{\ln T}, \quad T \rightarrow \infty.$$

*Remark 8.* At the same time the name *local equalities* is not quite exact. Namely, the elements of the matrix (3.10) standing in the  $2 - (k-1)$ th columns posses the property of internal non-locality (comp. part (b) of Remark 6).

#### 4. EXTENSION OF NON-LOCAL EQUALITIES (2.1) BY MEANS OF LOCAL EQUALITIES (3.10)

Next, adding local equalities (3.10) to non-local ones (2.1) we obtain the following

**Corollary.**

$$\begin{aligned}
 (4.1) \quad & g = \int_T^{\widehat{1}_{T+g}} \prod_0^0 \\
 & \parallel \\
 & \int_T^{\widehat{2}_{T+g}} \prod_0^1 = \int_T^{\widehat{1}_{T+g}} \prod_0^0 \\
 & \parallel \\
 & \int_T^{\widehat{3}_{T+g}} \prod_0^2 = \int_T^{\widehat{2}_{T+g}} \prod_0^0 = \int_T^{\widehat{1}_{T+g}} \prod_0^1 \\
 & \parallel \\
 & \vdots \\
 & \parallel \\
 & \int_T^{\widehat{k-1}_{T+g}} \prod_0^{k-2} = \int_T^{\widehat{k-2}_{T+g}} \prod_0^0 = \int_T^{\widehat{k-3}_{T+g}} \prod_0^1 = \cdots = \int_T^{\widehat{1}_{T+g}} \prod_0^{k-3} \\
 & \parallel \\
 & \int_T^{\widehat{k}_{T+g}} \prod_0^{k-1} = \int_T^{\widehat{k-1}_{T+g}} \prod_0^0 = \int_T^{\widehat{k-2}_{T+g}} \prod_0^1 = \cdots = \int_T^{\widehat{2}_{T+g}} \prod_0^{k-3} = \int_T^{\widehat{1}_{T+g}} \prod_0^{k-2}.
 \end{aligned}$$

*Remark 9.* Since (see (1.5))

$$\begin{aligned}
 (4.2) \quad & \int_T^{\widehat{p-s}_{T+g}} \prod_{r=0}^{s-1} \tilde{Z}^2[\varphi_1^r(t)] dt = \\
 & = \int_T^{\widehat{p-s}_{T+g}} \prod_{r=0}^{s-1} \frac{|\zeta(\frac{1}{2} + i\varphi_1^r(t))|^2}{\omega[\varphi_1^r(t)]} dt = g
 \end{aligned}$$

then we have the following: the set of equalities (4.1) is a kind of set of constraints on complicated behaviour of the function

$$\zeta\left(\frac{1}{2} + it\right), \quad t \rightarrow \infty.$$

*Remark 10.* Let us denote by

$$S(T, g)$$

the complete set of equalities-constraints contained in the matrix (4.1) for every fixed

$$[T, g] : T \in (T_0[\varphi_1, g], +\infty), \quad g \in (0, \infty).$$



Since every integral (energy) from the matrix (4.1) is invariant under the set of translations

$$T \longrightarrow T'; \quad T, T' \in (T_0[\varphi_1, g], +\infty)$$

then we call the set  $S(T, g)$  the *invariant set of equalities-constraints* for every fixed  $g \in (0, +\infty)$ .

#### APPENDIX A.

In this part we give two examples on the properties of finite additivity and of finite multiplicativity in the set of reversely iterated integrals (energies).

**A.1. On unbounded division of reversely iterated integral (energy) on equal parts.** Since for every fixed

$$[g, N] : \quad g \in (0, +\infty), \quad N \in \mathbb{N}; \quad g = o\left(\frac{T}{\ln T}\right)$$

(see (1.1)) we have that

$$g = N\delta(N)$$

then (see [4], (3.1), (3.3))

$$(A.1) \quad \int_k^{\widehat{T+N\delta}} \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt = N \int_k^{\widehat{T+\delta}} \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt.$$

Next, we have

$$\int_k^{\widehat{T+n\delta}} = n\delta, \quad \int_k^{\widehat{T+(n-1)\delta}} = (n-1)\delta \Rightarrow \int_k^{\widehat{T+n\delta}} = \delta, \\ n = 1, \dots, N,$$

i.e.

$$(A.2) \quad \int_k^{\widehat{T+n\delta}} = \int_k^{\widehat{T+\delta}}, \quad n = 1, \dots, N.$$

*Remark 11.* The formulae (A.1), (A.2) given a simultaneous ( $k = 1, \dots, k_0$ ) unbounded division of the reversely iterated integrals (energies) into equal parts.

**A.2. A chain of integrals.** If we use finite case of the complete multiplicativity (see [4], (4.1)) together with the formulae

$$\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos \omega x dx = e^{-\omega^2/2}, \quad \int_0^\infty e^{-x^2/2} dx = \sqrt{\frac{\pi}{2}},$$

then we obtain the following result.

*Remark 12.* The following holds true

$$\int_{(\mathbb{R}_0^+)^n} \left\{ \int_{\mathbb{R}_0^+} \prod_{l=1}^n \cos(\omega_l x_l) \times \right. \\ \left. \times \left[ \int_k^{\widehat{T+\prod \exp(-x_l^2/2)}} \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt \right] d\omega_1 \dots d\omega_n \right\} dx_1 \dots dx_n = \left(\frac{\pi}{2}\right)^n$$

for every fixed  $n \in \mathbb{N}$ .

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